TWO METHODS OF APPROXIMATE DESCRIPTION OF STEADY-STATE MOTIONS OF A VISCOUS INCOMPRESSIBLE LIQUID WITH A FREE BOUNDARY

V. V. Pukhnachev

UDC 532.516

In this paper waves on the surface of a viscous incompressible liquid are investigated in a linear approximation. It is shown that the linear theory gives the principal term of the solution of the problem of steady-state two-dimensional waves of small amplitude in an exact formulation. Subsequently a three-dimensional steady-state motion of a viscous liquid with high surface tension in a vessel is considered. In the first approximation the free boundary is determined as a minimum surface in a field of gravity. The velocity field is found from the solution of the Navier–Stokes equations.

1. Linear Approximation in the Theory of Surface Waves. The description of wave motions of a viscous liquid leads to a necessity of solving problems for the Navier–Stokes equations with an unknown boundary. Such problems at the present time are insufficiently studied (for the state of the problem see [1] and the bibliography given there and also [2, 3]). There exists a number of approximate models of surface waves in a viscous liquid. Historically the first of them was the linear theory of waves (Stokes [4]). This theory was developed in the investigations of Lamb [5], L. N. Sretenskii [6], and other authors.

As far as the author knows, up to now there is no answer to the question about the closeness of the solution of the wave problem in the exact formulation (as a problem with a free boundary for the Navier–Stokes equations) and in an approximation of the linear theory. Here this question is considered in the particular case of two-dimensional steady-state waves. In addition, the analysis is confined to an investigation of periodic wave motions of the forced-vibrations type. Examples of such motions are: a motion in a strip whose upper boundary is free, while the lower (the bottom) is a rigid straight-lined wall, with inflows and outflows of the liquid arranged periodically on it; steady-state gravity waves above an inclined periodic bottom; a motion excited by a periodic passing pressure wave or a tangential stress applied to the free surface.

With each of these flows we can associate a parameter which is proportional to the magnitude of the external action (the power of the sources, the angle of inclination of the mean line of the bottom to the horizon, the amplitude of the passing wave) and then consider the linear approximation with respect to this parameter. We shall consider the estimate of the error of the linear approximation to the solution of the problem in the exact formulation for small values of the parameter.

Below, for the sake of being definite, we consider the problem of periodic motion in a strip with sources and sinks distributed over the bottom. The mathematical formulation of the problem is as follows. It is required to find twice continuously differentiable functions $\mathbf{v}(x_1, x_2)$, (x_1) and a continuously differentiable function $p(x_1, x_2)$ which satisfy the relationships

$$\Delta \mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0 \tag{1.1}$$

within the strip $-\infty < x_1 < \infty$, $-1 < x_2 < f(x_1)$;

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 126–134, September-October, 1972. Original article submitted May 3, 1972.

$$\mathbf{v}(x_1 + l, x_2) \equiv \mathbf{v}(x_1, x_2), \quad p(x_1 + l, x_2) \equiv p(x_1, x_2), \quad f(x_1 + l) \equiv f(x_1) \tag{1.2}$$

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot T \cdot \boldsymbol{\tau} = 0 \quad \text{for} \quad x_2 = f(x_1) \tag{1.3}$$

$$\left(\frac{f'}{\sqrt{1+f'^2}}\right)' - \lambda f = \mu \mathbf{n} \cdot T \cdot \mathbf{n} \quad \text{for} \quad x_2 = f(x_1) \tag{1.4}$$

$$\int_0^l f\, dx_1 = 0 \tag{1.5}$$

$$\mathbf{v} = \varepsilon \mathbf{a}(x_1) \quad \text{for} \quad x_2 = -1 \tag{1.6}$$

Here $x_2 = f(x_1)$ is the equation of the free surface, $f' \equiv df/dx_1$, $\mathbf{n}$ and $\boldsymbol{\tau}$ are the base vectors of the normal and the tangent to the free surface; T is the stress tensor with the elements $T_{ij} = -p\delta_{ij} + \partial v_i/\partial x_j + \partial v_j/\partial x_i$, $i = 1,2$; $\mathbf{v}$ is the velocity vector, $p + \mu^{-1}\lambda x_2$ is the pressure.

The relationships (1.1)–(1.6) are written in terms of dimensionless variables; distances are referred to the mean depth of liquid h, velocities are referred to νh^{-1} (ν is the coefficient of kinematic viscosity), pressure is referred to $\rho\nu^2 h^{-2}$ (ρ is the density of liquid); $\lambda = \rho g h^2 \sigma^{-1}$, $\mu = \rho\nu^2(\sigma h)^{-1}$ are dimensionless positive parameters (g is the acceleration due to gravity; σ is the coefficient of surface tension); ε is a dimensionless positive parameter which in the following is assumed to be small.

The condition (1.3) signifies the absence of flow of the liquid and tangential stress on the free surface. According to the condition (1.4), the normal stress on the free boundary is equal to the surface pressure. The condition (1.5) shows that the dimensionless mean depth of the liquid is unity. In the condition (1.6) the vector function $\mathbf{a}$, which specifies the velocity on the bottom (for $x_2 = -1$), is assumed to be l-periodic with the components a_1, a_2 from the Holder class $C^{2+\alpha}[0, l]$ and such that

$$\int_0^l a_2\, dx_1 = 0 \tag{1.7}$$

The condition (1.7) is necessary for the agreement of the boundary conditions (1.6) and the first equation of (1.3) with the equation of continuity. In order to eliminate a possibility of contact of the free surface with the bottom, we stipulate yet that the inequality

$$|f| \leqslant \delta < 1 \quad (\delta = \text{const} > 0)$$

be fulfilled.

The existence of a unique solution of the problem (1.1)-(1.6) for small ε follows from the results of [3]. We note that introduction of the surface tension into the boundary condition (1.4) on the free boundary is essential: precisely this allows the solution of the problem to be reduced to the finding of a fixed point of a certain continuous operator [3].

We now formulate the equations of a linear approximation in the problem (1.1)–(1.6). For this we discard the nonlinear terms in Eqs. (1.1), while the boundary conditions (1.3), (1.4) are applied to the nonperturbed free boundary $x_2 = 0$. Denoting

$$\mathbf{v} = \varepsilon \mathbf{U}, \quad p = \varepsilon Q, \quad f = \varepsilon F$$

we arrive at the following linear boundary value problem with a fixed boundary:

$$\Delta \mathbf{U} - \nabla Q = 0, \quad \nabla \cdot \mathbf{U} = 0 \tag{1.8}$$

within the strip $-\infty < x_1 < \infty$, $-1 < x_2 < 0$;

$$\mathbf{U}(x_1 + l, x_2) \equiv \mathbf{U}(x_1, x_2), \quad Q(x_1 + l, x_2) \equiv Q(x_1, x_2), \quad F(x_1 + l) \equiv F(x_1) \tag{1.9}$$

$$U_2 = 0, \quad \partial U_1/\partial x_2 = 0 \quad \text{for} \quad x_2 = 0 \tag{1.10}$$

$$F'' - \lambda F = \mu(-Q + 2\partial U_2/\partial x_2) \quad \text{for} \quad x_2 = 0 \tag{1.11}$$

$$\int_0^l F\, dx_1 = 0 \tag{1.12}$$

$$\mathbf{U} = \mathbf{a}(x_1) \quad \text{for} \quad x_2 = -1 \tag{1.13}$$

The Stokes system (1.8) has a unique solution which satisfies the conditions (1.10), (1.13) and the first two conditions (1.9). If this solution is found, the function F is uniquely determined from the differential equation (1.11) and the conditions (1.12) and the last condition (1.9). According to the notions of the linear theory $x_2 = \varepsilon F(x_1)$ is the equation of the free boundary "in the first approximation."

2. Estimate of the Error of the First Approximation. We shall show that the function $\varepsilon F(x_1)$ determined from the linear problem (1.8)-(1.13) gives the principal term of the asymptotic expression, for $\varepsilon \to 0$, of the function $f(x_1; \varepsilon)$ which determines the form of the free boundary in the problem (1.1)-(1.6). A comparison of the velocity and pressure fields in the problem (1.1)-(1.6) and its linearization in terms of the variables $x_1 x_2$ is difficult, since the functions $\mathbf{v}$, p and $\mathbf{U}$, Q have different domains of definition. However, it is possible to map the region of flow in the problem (1.1)-(1.6) onto the strip $-1 < x_2 < 0$ and compare the solution of this problem in deformed coordinates with the solution of the problem (1.8)-(1.13). For this we go over to the new independent variables

$$\xi_1 = x_1, \quad \xi_2 = \frac{x_2 - f(x_1)}{1 + f(x_1)} \tag{2.1}$$

The system (1.1) for this is transformed into

$$\begin{gathered}
\frac{\partial_2 u_1}{\partial \xi_1^2} - \frac{2(1+\xi_2) f'}{1+f} \frac{\partial^2 U_1}{\partial \xi_1 \partial \xi_2} + \frac{1+(1+\xi_2)^2 f'^2}{(1+f)^2} \frac{\partial^2 U_1}{\partial \xi_2^2} + \left\{ \frac{(1+\xi_2)[2f'^2 - (1+f) f'']}{(1+f)^2} + \frac{(1+\xi_2) f' u_1}{1+f} - \frac{u_2}{1+f} \right\} \frac{\partial u_1}{\partial \xi_2} - \\
- u_1 \frac{\partial u_1}{\partial \xi_1} - \frac{\partial q}{\partial \xi_1} + \frac{(1+\xi_2) f'}{1+f} \frac{\partial q}{\partial \xi_2} = 0 \\
\frac{\partial^2 u_2}{\partial \xi_1^2} - \frac{2(1+\xi_2) f'}{1+f} \frac{\partial^2 u_2}{\partial \xi_1 \partial \xi_2} + \frac{1+(1+\xi_2)^2 f'^2}{(1+f)^2} \frac{\partial^2 u_2}{\partial \xi_2^2} + \left\{ \frac{(1+\xi_2)[2f'^2 - (1+f) f'']}{(1+f)^2} + \frac{(1+\xi_2) f' u_1}{1+f} - \frac{u_2}{1+f} \right\} \frac{\partial u^2}{\partial \xi_2} - \\
- u_1 \frac{\partial u_2}{\partial \xi_1} - \frac{1}{1+f} \frac{\partial q}{\partial \xi_2} = 0 \\
\frac{\partial u_1}{\partial \xi_1} - \frac{(1+\xi_2) f'}{1+f} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{1+f} \frac{\partial u_2}{\partial \xi_2} = 0
\end{gathered} \tag{2.2}$$

$$(u_1(\xi_1, \xi_2) = v_1(x_1, x_2), \ u_2(\xi_1, \xi_2) = v_2(x_1, x_2), \ q(\xi_1, \xi_2) = p(x_1, x_2); \ f(x_1) = f(\xi_1), \ f' = df/d\xi)$$

The boundary conditions (1.2)-(1.6) generate the following boundary conditions for the system (2.2):

$$\begin{gathered}
u_1(\xi_1 + l, \xi_2) \equiv u_1(\xi_1, \xi_2), \ u_2(\xi_1 + l, \xi_2) \equiv u_2(\xi_1, \xi_2), \ q(\xi_1 + l, \xi_2) \equiv \\
\equiv q(\xi_1, \xi_2) \ u_1 = \varepsilon a_1(\xi_1), \ u_2 = \varepsilon a_2(\xi_1) \quad \text{for } \xi_2 = -1 \\
\frac{1+f'^2}{1+f} \frac{\partial u_1}{\partial \xi_2} - 2f' \frac{\partial u_1}{\partial \xi_1} + (1 - f'^2) \frac{\partial u_2}{\partial \xi_1} + \frac{f'(1+f'^2)}{1+f} \frac{\partial u_2}{\partial \xi_2} = 0 \\
u_2 - f' u_1 = 0 \quad \text{for} \quad \xi_2 = 0
\end{gathered} \tag{2.3}$$

$$\frac{f''}{(1+f'^2)^{3/2}} - \lambda f = \mu \left[-(1+f'^2) q + \frac{2(1+f'^2)}{1+f} \frac{\partial u_2}{\partial \xi_2} + 2f'^2 \frac{\partial u_1}{\partial \xi_1} - \frac{2f'(1+f'^2)}{1+f} \frac{\partial u_1}{\partial \xi_2} - 2f' \frac{\partial u_2}{\partial \xi_2} \right] \quad \text{for} \quad \xi_2 = 0 \tag{2.4}$$

$$f(\xi_1 + l) \equiv f(\xi_1), \quad \int_0^l f(\xi_1) \, d\xi_1 = 0 \tag{2.5}$$

Proposition 2.1. For $\varepsilon \to 0$ the following estimates apply:

$$\begin{gathered}
|f(\xi_1; \varepsilon) - \varepsilon F(\xi_1)|_{3+\alpha, [0, l]} = O(\varepsilon^2) \\
|\mathbf{u}(\xi_1, \xi_2; \varepsilon) - \varepsilon \mathbf{U}(\xi_1, \xi_2)|_{2+\alpha, \Pi} = O(\varepsilon^2) \\
|q(\xi_1, \xi_2; \varepsilon) - \varepsilon Q(\xi_1, \xi_2)|_{1+\alpha, \Pi} = O(\varepsilon^2)
\end{gathered} \tag{2.6}$$

Here Π denotes the rectangle $0 \le \xi_1 \le l$, $-1 \le \xi_2 \le 0$; $\mathbf{u}$ is a vector with the components u_1, u_2; notation of the form $f = f(\xi_1; \varepsilon)$ is used to emphasize the dependence of the sought quantities on the parameter ε. If $\varphi(x) \in C^{m+\alpha}(\Omega)$, where $m \ge 0$ is an integer, Ω is a closed bounded region, then $|\varphi|_{m+\alpha, \Omega}$ denotes the norm of φ in $C^{m+\alpha}$.

The proof of Proposition 2.1 is based on the following a priori estimate of the solution of the problem (2.2)-(2.5):

$$|\mathbf{u}|_{2+\alpha, \pi} + |\nabla q|_{\alpha, \pi} + |f|_{3+\alpha, [0, l]} \leqslant C_1 \varepsilon |a|_{2+\alpha, [0, l]} \tag{2.7}$$

which is valid for a fixed a, if $\varepsilon \le \varepsilon_0$ and ε_0 is sufficiently small (C_k, $k = 1, 2, \ldots$ here and subsequently denotes positive constants). The estimate (2.7) in essence follows from the results [3] (in that work it was assumed that $\lambda = 0$, which is not essential; in boundary condition (1.6) the parameter ε was absent, but in spite of that the given velocity on the bottom was assumed to be small).

We introduce the functions $\mathbf{w} = \mathbf{u} - \varepsilon \mathbf{U}$, $r = q - \varepsilon Q$ into the analysis. On the basis of (2.2), (1.8) these functions satisfy in Π the system of equations

$$\Delta \mathbf{w} - \nabla r = \psi, \quad \nabla \cdot \mathbf{w} = \varphi \tag{2.8}$$

where Δ and ∇ are the Laplacian and the gradient with respect to the variables ξ_1, ξ_2, while ψ, φ are certain known functions of ξ_1, ξ_2 which are expressed in terms of $\mathbf{u}$, q, f and their derivatives. Essential are the following inequalities which are satisfied by the functions ψ, φ for $\varepsilon \to 0$:

$$|\varphi|_{1+\alpha,\Pi}\leqslant C_2\varepsilon^2,\quad |\psi|_{\alpha,\Pi}\leqslant C_2\varepsilon^2 \tag{2.9}$$

Both inequalities (2.9) are proved identically. We confine ourselves to the first of them. From (2.2), (1.8) we find

$$\varphi=\frac{(1+\xi_2)f'}{1+f}\frac{\partial u_1}{\partial\xi_2}+\frac{f}{1+f}\frac{\partial u_2}{\partial\xi_2} \tag{2.10}$$

Since $\mathbf{u}\in C^{2+\alpha}$ (II), $f\in C^{3+\alpha}[0,l]$ [3], we have $\varphi\in C^{1+\alpha}$ (II). The estimate (2.9) follows directly from the definition of φ (2.10) and the inequality (2.7) (since $\mathbf{a}$ is fixed, $|\mathbf{a}|_{2+\alpha,[0,l]}$ is included in the value of the constant C_2).

Proceeding from the conditions (1.9), (1.10), (1.13) and (2.3) it is not difficult to obtain boundary conditions for the functions $\mathbf{w}$, r. They have the form

$$\begin{gathered}w_2=\chi,\ \partial w_1/\partial\xi_2=\omega\quad\text{for}\quad\xi_2=0\\ \mathbf{w}(\xi_1+l,\xi_2)\equiv\mathbf{w}(\xi_1,\xi_2),\ q(\xi_1+l,\xi_2)\equiv q(\xi_1,\xi_2)\\ \mathbf{w}=0\quad\text{for}\quad\xi_2=-1\end{gathered} \tag{2.11}$$

where

$$\begin{gathered}\chi=f'u_1|_{\xi_2=0},\quad\omega=-f''(1+f)(1-f'^2)u_1+\\ +f'(1+f)(1+f'^2)\partial u_1/\partial\xi_1-f'(1+f'^2)\partial u_2/\partial\xi_2|_{\xi_2=0}\end{gathered} \tag{2.12}$$

The functions χ, ω belong to the classes $C^{2+\alpha}[0,l]$, $C^{1+\alpha}[0,l]$, respectively, and in view of (2.7) and (2.12) admit the estimates

$$|\chi|_{2+\alpha,[0,l]}\leqslant C_3\varepsilon^2,\quad|\omega|_{1+\alpha,[0,l]}\leqslant C_3\varepsilon^2 \tag{2.13}$$

The subsequent reasoning is based on applying to the boundary value problem (2.8), (2.11) the a priori estimates of the solutions of systems which are elliptic according to Douglis and Nirenberg [7]. The system (2.8) is a nonhomogeneous Stokes system; it is elliptic according to Douglis and Nirenberg [8]. The boundary conditions (2.11) for (2.8) satisfy the additionality condition formulated in [9, 8]. This guarantees the presence of estimates of $\mathbf{w}$, r that are exact in the limit in the Holder norms. Applying the results [9, 10] to the problem (2.8), (2.11) and using the inequalities (2.9), (2.13), we obtain the required estimate

$$|\mathbf{w}|_{2+\alpha,\Pi}+|\nabla r|_{\alpha,\Pi}\leqslant C_4\varepsilon^2 \tag{2.14}$$

for $\varepsilon\leq\varepsilon_0$. The absence of a term of the form $C|\mathbf{w}|_0$ from the right side is explained by the uniqueness theorem for the problem (2.8), (2.11) [3]: if $\psi=0$, $\varphi=\chi=\omega=0$, then $\mathbf{w}=0$, r = const. From the definition $\mathbf{w}=\mathbf{u}-\varepsilon\mathbf{U}$, $r=q-\varepsilon Q$ and the inequality (2.14) follows the correctness of the second estimate of (2.6).

The inequality (2.14) also signifies that

$$r(\xi_1,\xi_2;\varepsilon)=\gamma(\xi_1,\xi_2;\varepsilon)+K(\varepsilon) \tag{2.15}$$

where $|\gamma(\xi_1,\xi_2;\varepsilon)|_{1+\alpha,\Pi}=O(\varepsilon^2)$ for $\varepsilon\to0$, while K depends only on ε. It is desirable to show that $K(\varepsilon)=O(\varepsilon^2)$ for $\varepsilon\to0$. We transform Eq. (2.4), having substituted in it instead of the derivatives of u_2 for $\xi_2=0$ their expressions from the third equation (2.2) and the last relationship (2.3). We obtain

$$\left(\frac{f'}{\sqrt{1+f'^2}}\right)'-\lambda f=\mu\left\{-(1+f^2)q-2\frac{\partial}{\partial\xi_1}[(1+f'^2)u_1]+2f'f''u_1\right\}\Big|_{\xi_2=0} \tag{2.16}$$

Analogously the relationship (1.11) is transformed into

$$F''-\lambda F=\mu(-Q-2\partial U_1/\partial\xi_1)|_{\xi_2=0} \tag{2.17}$$

Integrating (2.16) from 0 to l and using (2.5) and the periodicity of u_1 with respect to ξ_1, we find

$$\int_0^l[(1+f'^2)q(\xi_1,0)-2f'f''u_1(\xi_1,0)]\,d\xi_1=0 \tag{2.18}$$

We note that for a given f Eqs. (2.2) and the boundary conditions (2.3) determine q with accuracy up to a constant summand. The relationship (2.18) allows us to eliminate arbitrariness in the determination of q. Analogously from (2.16), (1.9) and (1.12) we have the relationship

$$\int_0^l Q(\xi_1,0)\,d\xi_1=0 \tag{2.19}$$

which enables us to uniquely determine the function Q in the solution of the problem (1.8)-(1.10), (1.13). Using (2.18), (2.19), (2.7), (2.14), (2.15), we conclude that for $\varepsilon\to0$ the third estimate of (2.6) is valid.

The results thus obtained are then used for the proof of the remaining (first) inequality (2.6). We denote $b = f - \varepsilon F$. Subtracting Eq. (2.17), multiplied by ε, from (2.16) we obtain a differential equation for b:

$$b'' - \lambda b = \tau \tag{2.20}$$

with the right side

$$\tau = f''[1 - (1 + f'^2)^{-3/2}] - \mu [r + 2\partial w_1 / \partial \xi_1 + + f'^2 (q + 2\partial u_1 / \partial \xi_1) + 2f'f''u_1]|_{\xi_2=0} \tag{2.21}$$

(here we have used the definition $r = q - \varepsilon Q$, $w = u - \varepsilon U$). In view of (2.5), (1.9), (1.12), the function b satisfies the conditions

$$b(\xi_1 + l) \equiv b(\xi_1), \quad \int_0^l b\, d\xi_1 = 0 \tag{2.22}$$

Applying the inequalities (2.7) and the second and third from (2.6) to the estimate of the right side of (2.21), we obtain

$$|\tau(\xi_1; \varepsilon)|_{1+\alpha,[0,l]} = O(\varepsilon^2) \tag{2.23}$$

for $\varepsilon \to 0$. In addition, l as a periodic function of b in view of (2.18), (2.19), (2.5) has a zero mean value over the period. Hence it follows that for $\varepsilon \leq \varepsilon_0$ the solution b of the problem (2.20), (2.22) exists and is unique. On the basis of (2.23) this solution for $\varepsilon \to 0$ admits the estimate $|b|_{3+\alpha, [0, l]} = O(\varepsilon^2)$. With this we have proved the first inequality of (2.6). The proof of Proposition 2.1 is completed.

Concluding, we note that analogs of Proposition 2.1 are valid in the problem mentioned in Sec. 1, which is concerned with waves above a periodic bottom, and the problem concerned with the interaction of a passing wave with the free surface. The approach to the justification of a linear model of two-dimensional waves on the surface of a viscous liquid, presented above, admits also a generalization to certain three-dimensional problems.

3. Three-Dimensional Stationary Flow of a Capillary Liquid in a Vessel. Let a viscous incompressible liquid fill in a region $G = \{x_1, x_2, x_3 : (x_1, x_2) \in \Omega,\ 0 < x_3 < f(x_1, x_2)\}$ and be in a steady-state motion. Here Ω is a bounded region of the x_1, x_2 plane with a sufficiently smooth boundary S. The motion is induced by sources and sinks with zero overall flow distributed on the bottom $\Sigma = \{x_1, x_2, x_3 : (x_1, x_2) \in \Omega, x_3 = 0\}$. The surface $\Gamma = \{x_1, x_2, x_3 : (x_1, x_2) \in \Omega,\ x_3 = f(x_1, x_2)\}$ is assumed to be free. The cylindrical surface $B = \{x_1, x_2, x_3 : (x_1, x_2) \in S,\ 0 < x_3 < f(x_1, x_2)\}$ is a rigid impermeable wall.

The equations of motion written in dimensionless variables have the form

$$\Delta \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p = 0, \quad \nabla \cdot \mathbf{v} = 0 \tag{3.1}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity vector, p is the pressure deviation from the hydrostatic pressure (it is assumed that the force of gravity acts in the direction opposite to the direction of the x_3 axis). Dimensionless variables are introduced in the same way as in Sec. 1, with the difference that now h is the diameter of the region Ω.

For the system (3.1) we set up the boundary conditions

$$\mathbf{v}|_\Sigma = \mathbf{a}(x_1, x_2), \quad \mathbf{v}|_B = 0 \tag{3.2}$$

$$\mathbf{v}|_\Gamma \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot T|_\Gamma \cdot \boldsymbol{\tau} = 0 \tag{3.3}$$

$$\nabla_2 \cdot \left[\frac{\nabla_2 f}{\sqrt{1 + |\nabla_2 f|^2}} \right] - \lambda f = \mu \mathbf{n} \cdot T \Big|_\Gamma \cdot \mathbf{n} \tag{3.4}$$

$$(1 + |\nabla_2 f|^2)^{-1/2} \frac{\partial f}{\partial N}\Big|_S = \cos\theta \tag{3.5}$$

Here **n** is the unit vector of the outer normal and $\boldsymbol{\tau}$ is an arbitrary unit vector lying in the tangent plane to the free surface; T is the stress tensor; ∇_2 is the two-dimensional gradient with respect to the variables x_1, x_2; $\nabla_2 \cdot \mathbf{e}$ is the divergence of the vector $\mathbf{e} = (e_1, e_2)$; the parameters λ and μ have been introduced in Sec. 1. Under the conditions (3.2) **a** is a given vector function of the Holder class $C^{2+\alpha}(\bar{\Omega})$, $0 < \alpha < 1$ ($\bar{\Omega}$ is the closure of Ω) which is finite in Ω and such that

$$\int_\Omega a_3\, dx_1\, dx_2 = 0 \tag{3.6}$$

When writing (3.4) it is assumed that the right side is represented as a function of x_1, x_2. In (3.5) N is the direction of the outer normal to S, while θ is a boundary angle which is determined by the properties of the liquid and the material of the walls of the vessel. This condition plays the part of a boundary condition for the relationship (3.4), if the latter is treated, for a given right side, as an elliptic equation for $f(x_1, x_2)$.

The theorem of existence and uniqueness of the solution of the problem (3.1)-(3.5) has not been proved. However, there are grounds for the assumption that this problem even for small $|\mathbf{a}|_{2+\alpha, \bar{\Omega}}$ has a single-parameter family of solutions. Consideration of a periodic analog of the problem [3] as well as the analysis of the approximation of the problem (3.1)-(3.5), proposed below, in the case where the parameter μ is small leads to such a conclusion. Therefore it is considered necessary to set up side by side with (3.1)-(3.5) yet another condition. From physical considerations, it is natural to specify the mean depth of liquid $\bar{f}$:

$$\int_{\Omega} (f - \bar{f})\, dx_1\, dx_2 = 0 \tag{3.7}$$

The approximate solution of the problem (3.1)-(3.5), (3.7) for small μ is based on replacing, on the right side of (3.4), the function $\mu \mathbf{n} \cdot \mathrm{T}|_{\Gamma} \cdot \mathbf{n}$ by a constant. Then the relationships (3.4), (3.5), (3.7) form a closed system for the determination of f. Further it turns out that for a fixed f the problem (3.1)-(3.3) always has a solution. If $|\mathbf{a}|_{2+\alpha, \bar{\Omega}}$ is small, then the field of velocities is uniquely determined, while the pressure is determined with accuracy up to a constant, $p = p^* + \lambda\mu^{-1} C$, where p^* is fixed, for example, by the condition $p^*(x_1^\circ, x_2^\circ, 0) = 0$ at a certain point $(x_1^\circ, x_2^\circ, 0) \in \Sigma$. Having denoted by T^* the stress tensor corresponding to the field $\mathbf{v}$, p^*, we obtain $\mu\mathbf{n}\cdot T^*|_\Gamma\cdot\mathbf{n} = -\lambda C + \mu\mathbf{n}\cdot T|_\Gamma\cdot\mathbf{n}$. The hypothesis consists of the fact that for the solution of the problem (3.1)-(3.5), (3.7) the tensor T^* regularly depends on μ for $\mu \to 0$.

Putting $\mu\mathbf{n}\cdot \mathrm{T}|_\Gamma \cdot \mathbf{n} = -\lambda C$ = const on the right side and making the substitution

$$f = y + C \tag{3.8}$$

we arrive at the following problem for the determination of the function $y(x_1, x_2)$:

$$\nabla_2 \left[\frac{\nabla_2 y}{\sqrt{1 + |\nabla_2 y|^2}} \right] - \lambda y = 0 \tag{3.9}$$

$$(1 + |\nabla_2 y|^2)^{-1/2} \frac{\partial y}{\partial N}\bigg|_S = \cos\theta \tag{3.10}$$

The problem (3.9), (3.10) is a nonlinear boundary value problem for an elliptical equation of the minimal-surfaces type. Its solution describes a form of equilibrium of a capillary liquid in a field of gravity. The theory of such problems up to the recent time has been insufficiently developed. There are, however, a number of important particular cases where the problem (3.9), (3.10) permits effective investigation. In a one-dimensional case, when Ω is the strip $0 < x_1 < h$ and y does not depend on x_2, this problem is solved explicitly. If Ω is a circle $(x_1^2 + x_2^2)^{1/2} = r < h$, then we can seek axisymmetric solutions $y = y(r)$. A number of investigations (see [11] and the bibliography found there) are devoted to the numerical solution and qualitative analysis of the problem (3.9), (3.10). The existence of axisymmetric solutions is proved in [12].

In the case where Ω is an arbitrary bounded region with a boundary of class $C^{2+\alpha}$ while the boundary angle θ is close to $\pi/2$, we can find a small solution of the problem (3.9), (3.10) (we note that for $\theta = \pi/2$ the only solution is $y = 0$). This solution exists, is unique, and is found by the method of successive approximations. In [13] the problem (3.9), (3.10) was investigated for an arbitrary region Ω but for large values of the parameter $\lambda > 0$. The authors [13] proved unique solvability of the problem and obtained an asymptotic expansion of the solution for $\lambda \to \infty$.

Not long ago there appeared a paper by N. N. Ural'tseva [14] in which problems are studied for non-uniformly elliptic equations which are more general than (3.9). The region Ω in [14] is assumed to be convex, while the boundary condition is assumed to be homogeneous. This corresponds to $\theta = \pi/2$ in (3.10). With these assumptions we consider Eq. (3.9) with a nonzero right side (this signifies that on the surface of the liquid the pressure is distributed according to a given law). From the results [14] it follows that there exists a unique solution of this equation which satisfies the condition $\partial y/\partial N = 0$ on S.

Let $y(x_1, x_2)$ be a certain solution of the problem (3.9), (3.10). To determine f from the given y, we have to find the constant C from (3.8). For this we substitute $y = f - C$ into (3.7), while the integral of y over the region is calculated, integrating by parts Eq. (3.9) and using the condition (3.10):

$$C = \bar{f} - \lambda^{-1} \varkappa \cos\theta \tag{3.11}$$

where $\varkappa$ is the ratio of the perimeter of the region Ω to its area.

We determine $f(x_1, x_2)$ from the relationships (3.8)-(3.11). Then the conditions (3.5), (3.7) are satisfied exactly, and (3.4) approximately, if μ is small. For a given $f \in C^{1+\alpha}(\bar{\Omega})$ we consider the boundary value problem (3.2), (3.3) for the Navier–Stokes equations (3.1). Its solution admits an a priori estimate of the norm of **v** in a Sobolev vector space $\mathbf{W}^1_2(G)$. This allows us, following the reasoning [3], to prove that the problem (3.1)-(3.4) always has at least one generalized solution. The solution is unique if $|\mathbf{a}|_{2+\alpha,\bar{\Omega}}$ is sufficiently small. The functions **v**, p are infinitely differentiable within the open region G. If $f \in C^{3+\alpha}(\bar{\Omega})$, then $\mathbf{v} \in C^{2+\alpha}(\bar{G}')$, $\nabla p \in C^{\alpha}(\bar{G}')$ for any closed subregion G' of the region $\bar{G}$ which does not contain points of intersection of the free surface and the bottom with the side boundary G.

The condition of applicability of this approximation in the problem (3.1)-(3.5), (3.7) requires that for other fixed parameters the coefficient of surface tension σ be sufficiently large. We note that for water $\sigma = 72.5$ g/sec^2, $\nu = 0.01$ cm^2/sec at 20°C, $\rho = 1$ g/cm^3. Consequently $\mu < 10^{-6}$ for h > 1.4 cm. It would, however, be possible to consider σ, and together with it μ, as fixed, but consider a slow motion. This corresponds to a small vector function $\mathbf{a}(x_1, x_2)$. Assuming that $\mu \mathbf{n} \cdot T^*|_\Gamma \cdot \mathbf{n} \to 0$ for $\mathbf{a} \to 0$, we arrive at the scheme of approximate solution of the problem of flow in a vessel presented above.

Finally, analogous considerations apparently are applicable to a problem concerned with a flow in a deep vessel. Let μ, λ, θ, **a**, Ω be fixed and $f \to \infty$. We assume that for this $T_{ij}^*|_\Gamma \to 0$, $\mathbf{n} \cdot T|_\Gamma \cdot \mathbf{n} \to$ const. This allows us, as before, to determine approximately the free boundary as a minimum surface in a field of gravity. The physical meaning of the assumption made consists of the fact that the shape of the free surface is only slightly influenced by sources and sinks located far away from it. The hypothesis about $T_{ij}^*|_\Gamma \to 0$ for $f \to \infty$ corresponds to the well-known principle of Saint-Venant in the theory of elasticity (see, for example, [15], where different versions of this principle are presented).

We consider an analog of the problem (3.2), (3.3) for the Stokes system $\Delta \mathbf{v} - \nabla p = 0$, $\nabla \cdot \mathbf{v} = 0$ in a plane case. Using the methods of the paper by Knowles [15], we can show that the elements of the tensor T* exponentially decrease, as x_3 increases, along any segment which is parallel to the x_3 axis and lies inside the region G. It can be hoped that a similar assertion is valid also for the solution of the nonlinear three-dimensional problem (3.1)-(3.3) at least in the case where **a** is small.

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